ON SPECIALIZATION BETWEEN FINITE TOPOLOGIES AND DIRECTED GRAPHS

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Abstract: In this work we associiate a digraph to a topology by means of the “specialization” relation between points in the topology. Reciiprocally, we associate a topology to each diigraph, taking the sets of vertices adjacent (in the digraph) to v, for all vertex v, as a subbasis of closed sets for the topology. And study some properties of finite topology and directed graphs by this relation.

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1. Introduction:

In our work [1][2][3] we study some subgraph of graph of binary relations such as graph of posets ,graph of preorders relations and finite topology and graph of acyclic relation such that we defined new specialization between themes and we fined some relations between these concepets and finite toplogy . Even though graphs are simple relational combinations, they can represent topological spaces, combiinatorial objects and many other mathematical combinations. Many concepts will be very useful from practical perspective when they abstractly represented by graphs and this represents the second reason for importance of graphs . Topology is an interesting and imporrtant field of mathematics The one to one correspondence between finite preorder relations and finite topologies with the same underlying set of points, and also between finite posets and finite T_0 topologies is well known. Then, the one to one correspondence between finite digraphs and topologies is easily deducible. In fact, Evans, Harary and Lynn [3] prove that “There is a 1-1 corresspondence between the labeled topologies with n points and the labeled transitive digraphs with n points”. They associate a transitive digraph $D(T)$ to a topology T with the same set of points as follows: “For two distinct points u and v of T , u will be adjacent to v in $D(T)$ provided u is in every neighborhood of v”. Reciprocally, “to each labeled transitive digraph D with n points, there corresponds a unique labeled topology $T(D)$, in which the basic open sets are the sets of poiints adjacent to v, for all points v”. In this research, we study the relations between finite topologies and digraphs in a different way. We associate a digraph to a topology by means of the “specialization” relation between points in a topological space: x is a specialization of y if and only if x is in the closure of $\{y\}$. Reciiprocally, we associate a topology T to each digraph D (not necessarily transitive) taking the sets of vertices adjacent to v in D, for all vertices v, as a subbasis of closed sets for the topology T . And we use these two asociations to make a more profound study of the relations between simple digraphs and their homologous topologies. We also extend the relation between these structures to the functions.

preseerving the structure between these classes of objects. We restrict the study to the particular bijective relation between finite acyclic transitive digraphs and T0 topologies. In this context we consider dual concepts to those used in the preceding section to obtain a minmum basis of open sets for the topology T and to prove that the set of the closures of the points in T is the minimum basis of open sets in the dual situation.

2. Finite directed graphs

By a digaraph we mean a pair (X, G) where X is a finite nonempty set and $G \subset X \times X - \{(x, x) : x \in X\}$, (so our digraph has no loops). The elements in X and G are called points and arcs respectively. For an arc (x, y) we wiill say that x is adjacent to y. In the followiing we will denote by xy an arc (x, y).

Definition 2.1[5]: A sequence $x_1 x_2 \dots x_q x_{q+1}, q \geq 2$, of distinct points, except $x_{q+1} = x_1$, is a cycle if $x_i x_{i+1} \in G$ for $i = 1, 2, \dots, q$. A digraph is acyclic if it has

no cycle. A digraph (X, G) is said to be totally diisconnected if $G = \emptyset$. The non existence of loops in a digraph requires the following correction in the standard concepts of transitivity and antisymmetry.

Definition 2.2[4]: A digraph (X, G) is said to be transitive if $xy, yz \in G,$ with $x \neq$ z, then $xz \in G$. A digraph (X, G) is said to be antiisymmetric

if $xy \in G,$ then $yx \notin G$.

Remake 2.3: We denote the set of digaraphs with a set of points X by G_X , the subset of transitve digraphs by G_X^T and the antisymmetric transitive digraphs by G_X^{TA} .

Proposition 2.4: Let (X, G) be a transitive digaraph. Then (X, G) is acyclic if and only if it is antisymmetric.

Proof: An acyclic digraph is antisymmetric because, if there are arcs xy and yx, then it must be the cycle xyx. Reciiprocally, if $x_1 x_2 \dots x_q x_{q+1}$ is a cycle then, by transitivity, x_1x_q and x_qx_1 are arcs, in contradiction with the antisymmetric property.

3. Finite topological spaces

Let X be a nonempty set whose elemenets we will call points. Then, a topology T on X is a set of subsets of X, including $\emptyset$ and X, that is closed under union and finite intersection and the couple (X,T) is a topologiical space on X [4]. The elements in T are called open sets and their complments closed sets. The largest topology on X, $T = P(X)$, is called the discrete topology. If (X,T) is a topological space and $A \subset X$, the closure of A is the minmum closed set that contains A, and we denote it by $\bar{A}_T$, or simply by $\bar{A}$ if there is no possible confusion. We also use some other standard topological concepts such as basis and subbasis of open or closed sets, neighborhood, connecetion,. In the following definition we describe some "separation properties" by means of conditions easily related to each other. In this context, we use E^d to denote the derived set of $E \subset X$.

Definition 3.1[4]: Let (X,T) be a topologiical space. Then we will say that (X,T) is T_0 if $\forall x \in X, \{x\}^d$ is a union of closed sets or, equivalently, if $\forall x,y \in X$, with $x \neq y$, then $\overline{\{x\}} \neq \overline{\{y\}}$. (X,T) is T_D if $\forall x \in X$, $\{x\}^d$ is a closed set . (X,T) is T_1 if $\forall x \in X$, $\{x\}^d = \emptyset$ or, equivalently, if $\forall x \in X, \overline{\{x\}} = \{x\}$.

Remark 3.2 It is well known that, in general, $T_1 \Rightarrow T_D \Rightarrow T_0$, and that the reciprocal is not true. A topologiical space (X,T) is finite if the set X is finite. In this case, the following result can be deduced from the above definitions.

Lemma 3.3 Let (X, T) be a finit topological space. Then

1. $T_D \Leftrightarrow T_0$.

2. $T_1 \Leftrightarrow T$ is the discrete topology.

"Other separation properties, of a genereal topological space, more restrictive than T_1 (such as $T_2, T_{2a}, T_3, T_{3a}, T_4, \ldots$) are equivalent, in a finite topological space, to T_1. In this way, T_0 is the only relevant separation property in a finite nondiscrete topological space."

4. Relations between topological spaces and digraphs

The "specialization" relation between points in a topological space was introduced by Alexandroff [4]. In this work we introduced the specialization in the other way.

Definition 4.1 Let (X,T) be a topological space. For each pair of points $x,y \in X$ we will say that x is a specialization of y if and only if $x \in \overline{\{y\}}$. The specialization relation is a preorder on X: it is obviously reflexive, and is transitiive because

$$x \in \overline{\{y\}} \Rightarrow \overline{\{x\}} \subset \overline{\{y\}} \text{ and } y \in \overline{\{z\}} \Rightarrow \overline{\{y\}} \subset \overline{\{z\}}$$

and so $x \in \overline{\{x\}} \subset \overline{\{z\}}$. But, in general, it is not antisymmetric, because in a topology, distinct points can have the same closure, in which case the points are related in both ways. This relatiion permits us to associate a digraph over X to each topological space on the set of points X. We denote the set of topologiical spaces over a same finite set of points X by T_X , and the set of T_0 topologcal spaces over X by T_X^0.

Proposition 4.2 Let $f : T_X \longrightarrow G_X$ be the function given by $f(X,T) = (X,G)$ where G is the set of arcs $G = \{xy : x \in \overline{\{y\}}, x \neq y\}$ Then

a) f is injective.

b) $f(X,T) = (X,G)$ is a transitive digraph.

c) f is not suprajective.

Proof: a) If T and T' are distinct topologies over X, then there exists at least a point x with distinct respeective closures, that is to say, $\overline{\{x\}}_T \neq \overline{\{y\}}_{T'}$. Then, there exists a point y with $y \in \overline{\{x\}}_T$ and $y \notin \overline{\{x\}}_{T'}$ (or vice versa) and so $yx \in G$ and $y \notin G'$ (or vice versa), then $f(X,T) \neq f(X,T')$.

b) It is a consequence of the transitivity of the specialization relation.

c) f is not suprajective because the nontransitive digraphs do not procced from any topology. It is also possible to associate a topological space over X to each digraph with set of points X by means of the following procedure.

Definition 4.3[6]: Let $g : G_X \to T_X$ be the funcition given by $g(X,G) = (X,T)$ where T is the topology over X generated by the subbasis of closed sets

$$G\downarrow = \{x\downarrow : x \in X\}, \text{ where } x\downarrow = \{y : yx \in G\} \cup \{x\}$$

We also use the notation $g(G) = T$. From definition a bove we get the following:

Proposition 4.4 With the notations as above we have

a) g is suprajective.

b) $g \circ f$ is the identity function on T_X .

c) g is not injective.

d) g does not preserve the inclusion relation between digraphs.

e) $f \circ g : G_X \to G_X$ is not the identity and nor does it preserve the inclusion.

Proof: a) For any topologiical space $(X,T) \in T_X$ we consider the digraph $(X,G) = f(X,T)$ and we will prove that $g(X,G) = (X,T)$. We have $G = \{xy : x \in \overline{\{y\}}, x \neq y\}$ and g(X, G) is the topological space (X, T 0) whose subbasis of closed sets is $G\downarrow = \{x\downarrow : x \in X\}$ where

$$x \downarrow= \{y : yx \in G, x \neq y\} \cup \{x\}$$
$$= \{y : y \in \overline{\{x\}}, x \neq y\} \cup \{x\} = \{x\}$$

so $G \downarrow= \{\{x\} : x \in X\}$ and, as this is the subbasis of closed sets of the topology T, we have $T = T'$.

b) $gof = I_{T_X}$ is a consequence of the construction in a).

Next we give counterexamples proving c), d) and e).

Example 4.5 : For each of the digraphs (X, G) on Figure 1 we construct:

a) the family $G \downarrow$ of unipoint adjacencies $x \downarrow$ (that we will take as a subbasis of closed sets for the topology $T = g(G)$),

b) the topology T given by its closed sets,

c) the family $T = \{\{x\}, x \in X\}$ of unipoint closures of T and

d) the digraphs $f(X, T)$, that we denote by G', given in Figure 2. In this way we have $f \circ g(G) = G'$ according to the scheme

$$G \to G \downarrow\to T \to \bar{T} \to G'$$

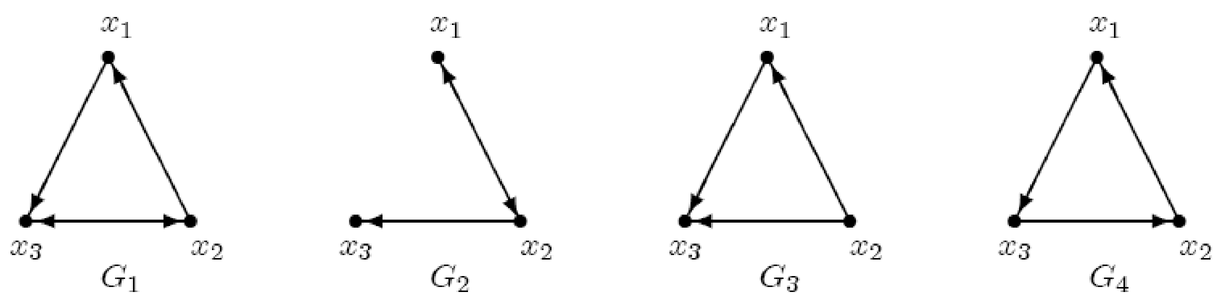

Figure 1

$G_1 \downarrow= \{x_1 \downarrow= \{x_1, x_2\},\ x_2 \downarrow= \{x_2, x_3\},\ x_3 \downarrow= X\}, T_1 = \{\{x_2\}, \{x_1, x_2\},$ $\{x_2, x_3\}, X, \emptyset\}$ and $T_1 = \{\{x_2\} = \{x_1, x_2\}, \{x_2\} = \{x_2\}, \{x_3\} = \{x_2, x_3\}\}$.

$G_2 \downarrow= \{x_1 \downarrow= \{x1, x_2\},\ x_2 \downarrow= \{x_1, x_2\},\ x_3 \downarrow= \{x_2, x_3\}\}, T_2 = T_1$ *and*

$$T_2 = T_1.$$

$G_3 \downarrow = \{x_1 \downarrow = \{x_1, x_2\},\ x_2 \downarrow = \{x_2\},\ x_3 \downarrow = X\},\ T_3 = \{\{x_2\}, \{x_1, x_2\}, X, \emptyset\}$

and $T3 = \{\{x_1\} = \{x_1, x_2\}, \{x_2\} = \{x_2\}, \{x_3\} = X\}$.

$G_4 \downarrow = \{x_1 \downarrow = \{x_1, x_2\},\ x_2 \downarrow = \{x_2, x_3\},\ x_3 \downarrow = \{x_1, x_3\}\}$, T_4 is the discrete topology and $T_4 = \{\{x_1\} = \{x_1\}, \{x_2\} = \{x_2\}, \{x_3\} = \{x_3\}\}$.

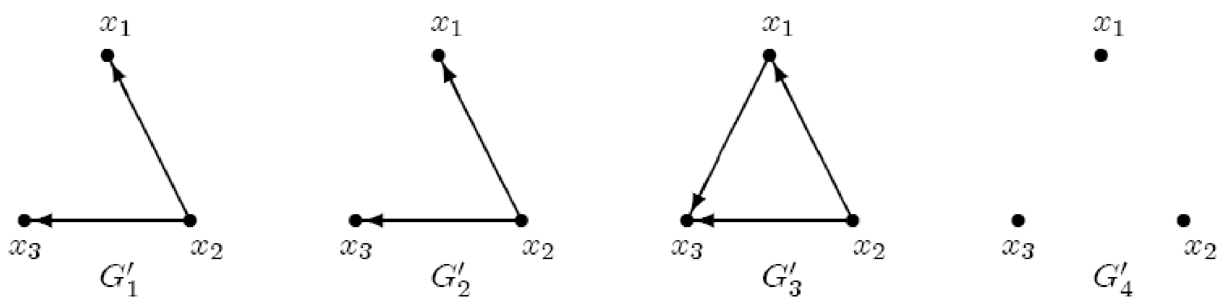

Figure 2

"Note that G_1 and G_2 are non comparable by the inclusion relation but they have the same image by g and by $f \circ g$, proving c) of Proposition 4.4 G_1 and G_3 verify $G_3 \subset G_1$ but $f \circ g(G_3) \supset f \circ g(G_1)$, proving e) of Proposition 4.4 G_1 and G_4 verify $G_4 \subset G_1$ and $T_4 \supset T_1$ and, consequently, $g(G_4) \supset g(G_1)$ and $f \circ g(G_4) \supset f \circ g(G_1)$, proving d) of Proposition 4.4 G_3 and G_4 are digraphs with "regular behaviour" under f and g as we shal see now.

The non-transtivity is the cause of these anomalies. In the example, only G_3 is transitive and for this one we have $f \circ g(G_3) = G_3$.

References

[1] Al'Dzhabri Kh.Sh., Rodionov V.I. , The graph of partial orders, Vestn. Udmurt. Univ. Mat. Mekh. Komp'yut. Nauki, 2013, issue 4, pp. 3-12 (in Russian). DOI: 10.20537/vm130401

[2] Al'Dzhabri Kh.Sh. ,The graph of reflexive-transitive relations and the graph of finite topologies, Vestn. Udmurt. Univ. Mat. Mekh. Komp'yut. Nauki, 2015, Vol. 25, issue 1, pp. 3-11 (in Russian). \\ DOI: 10.20537/vm150101

[3] Al' Dzhabri Kh.Sh., Rodionov V.I. ,The graph of acyclic digraphs, Vestn. Udmurt. Univ. Mat. Mekh. Komp'yut. Nauki, 2015, Vol. 25, issue 4, pp. 441-452 (in Russian). DOI: 10.20537/vm150401

[4] Alexandroff, P. S., Diskrete Rume, Matematiceskii Sbornik (N.S.) 2, pp. 501-518, (1937).

[5] Evans, J., Harary, F., Lynn, M., On the computer enumeration of finite topologies, Comm. ACM., 10, pp. 295-298, (1967).

[6] Grothendieck, A., Dieudonn´e, J.A., El´ementes de G´eometrie Alg´ebrique I, Springer-Verlag, Berlin, (1971).