The First Integral Method for Solving Two-Dimensional Reaction–Diffusion Brusselator System

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Abstract

In this paper, we studied non-stationary Reaction–Diffusion Brusselator systems in two-dimensional domain by using the first integral method and this method is based on the theory of commutative algebra. We obtained different types of exact solution by using types of variable transformations.

Keywords: Two-dimensional reaction–diffusion Brusselator system, the first integral method, exact solution.

1- Interdiction

In science, many important phenomena in various fields can be described by nonlinear partial differential equations. When these equations are analyzed, one of the most important question is the construction of the exact solution of these equations. The investigation of these solutions plays an important role in the study of nonlinear phenomena [2].

Recently, introduced a reliable and effective method called Feng's first integral method to look for travelling wave solutions of nonlinear partial differential equations. The basic idea of this method is to construct a first integral with polynomial coefficients of an explicit form to an equivalent autonomous planar system by using division theorem. This method in comparison with other methods has many advantages, it avoids a great deal of complicated and tedious calculation and provides exact and explicit travelling solution with nigh accuracy. Feng's first integral method can be used to construct the exact solution for some time fractional differential equation [3].

The Brusselator model, the nonlinear system of partial differential reaction-diffusion processes. This Brusselator equation, arises in the modeling of certain chemical model plays a substantial role in the study of cooperative processes reaction diffusion of chemical kinetics [9]. Reaction-Diffusion models are reaction-diffusion equations. One of such important reaction-diffusion equations is known as Brusselator system, which is used to describe mechanism of chemical reaction-diffusion with non-linear oscillations [10].

Reaction-diffusion systems are well known to self-organize into a variety of spatio-temporal patterns including, spots, stripes, as well as spatio-temporal chaos and uniform oscillations. Their existence in out-of-equilibrium states, connection to idealized chemical systems, and dependence on dimensional parameters, make them a good test bench for the study of general features of pattern generation and evolution. In particular, the dependence of these final states on the rate at which constituents are fed into the system (feed-rate) is of significant interest, since systems that can exchange matter reaction-diffusion systems represent proxies for high-level biological system environment. Depending on the value of the feed-rate, the and energy with the may asymptote into one of many states and thus the federate can be thought of playing the role of a natural control parameter [5].

2- The First Integral Method

The core structure of the first integral method is presented below [6,1,7].

Step1. Consider a general nonlinear partial differential equation in the form

$$E(u,\ u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0 \qquad (1)$$

To find the travelling wave solutions to Equation (1), we introduce the wave variable

$$\xi = x \mp ct, \tag{2}$$

so that

$$u(x,\ t) = u(\xi) \tag{3}$$

Based on this we use the following changes

$$\begin{aligned} &\frac{\partial}{\partial x}(\cdot) = \frac{d}{d\xi}(\cdot) \\ &\frac{\partial}{\partial t}(\cdot) = -c\frac{d}{d\xi}(\cdot) \\ &\frac{\partial^2}{\partial x^2}(\cdot) = \frac{d^2}{d\xi^2}(\cdot) \\ &\frac{\partial^2}{\partial t \partial x}(\cdot) = -c\frac{d^2}{d\xi^2}(\cdot) \end{aligned} \tag{4}$$

and so on for the other derivatives. Using (4) changes the PDE (1) to an ODE

$$H\left(u,\ \frac{du}{d\xi}, \frac{d^2u}{d\xi^2},\ \dots\right) = 0$$

(5)

where $u = u(\xi)$ is an unknown function, H is a polynomial in the variable u and its derivatives.

Step2. Suppose the solution of ODE (5) can be written as follows:

$$u(x,t) = f(\xi) \tag{6}$$

and furthermore, we introduce a new independent variable

$$X(\xi) = f(\xi),\ \ Y(\xi) = \frac{df(\xi)}{d\xi} \tag{7}$$

Step3. Under the conditions of Step 2**,** Equation (5) can be converted to a system of nonlinear ODEs as follows

$$\begin{aligned} X'(\xi) &= Y(\xi), \\ Y'(\xi) &= F\big(X(\xi), Y(\xi)\big) \end{aligned} \tag{8}$$

If we can find the integrals to Equation (8), then the general solutions to Equation (8) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is neither a systematic theory that can tell us how to find its first integrals, nor a logical way for telling us what these first integrals are. We will apply the so-called Division Theorem to obtain one first integral to Equation (8) which reduces Equation (5) to a first order integrable ODE. An exact solution to equation (1) is then obtained by solving this equation.

Division Theorem **[4]** Suppose that $P\ (w,\ z)$ and $Q\ (w,\ z)$ are polynomials in $C\ [w,\ z]$, and $P\ (w,\ z)$ is irreducible in $C\ [w,\ z]$. If $(w,\ z)$ vanishes at all zero points of $P\ (w,\ z)$, then there exists a polynomial $G\ (w,\ z)$ in $C\ [w,\ z]$ such that $Q\ (w,\ z) = P\ (w,\ z)G\ (w,\ z)$.

The Divisor Theorem follows immediately from the Hilbert–Nullstellensatz Theorem.

Hilbert–Nullstellensatz Theorem [4] Let K be a field and L be an algebraic closure of K . Then:

(i) Every ideal γ of $K[X_1,X_2\ ,\ldots,X_n]$ not containing 1 admits at least one zero in L^n.

(ii) Let $x\ (x_1\ ,x_2,\ldots,x_n)$ and $y\ (y_1,y_2,\ldots,y_n)$ be two elements of L^n. For the set of polynomials of $K[X_1,X_2\ ,\ldots,X_n]$ zero at x to be identical with the set of polynomials of $K[X_1,X_2\ ,\ldots,X_n]$ zero at y , it is necessary and sufficient that there exists a $K -$ automorphism S of L such that for $y_i = S(x_i)\ for\ 1 < i < n$.

(iii) For an ideal α of $K[X_1,X_2\ ,\ldots,X_n]$ to be maximal, it is necessary and sufficient that there exists an x in L^n such that α is the set of polynomials of $K[X_1,X_2\ ,\ldots,X_n]$ zero at x .

(iv) For a polynomial Q of $K[X_1,X_2\ ,\ldots,X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $K[X_1,X_2\ ,\ldots,X_n]$ it is necessary and sufficient that there exists an integer $m > 0$ such that $Q^m \in \gamma$.

3. Application

Here we illustrate the first integral equation for the two-dimensional reaction-diffusion Brusselator system with time dependent coefficients. Let is consider the two-dimensional reaction-diffusion Brusselator system formula [8]:

$$\frac{\partial u}{\partial t} = B + u^2\, v - (A+1)\, u + \alpha\, \nabla^2\, u$$
$$\frac{\partial v}{\partial t} = A\, u - u^2\, v + \alpha\, \nabla^2\, v$$

for $u(x,y,t)$ and $v(x,y,t)$ in a two-dimensional, where A, B and α are suitably given constants

We apply the first integral method presented above on reaction-diffusion Brusselator system

$$-c\frac{df}{d\xi} = B + f^2 g - (A+1)f + \alpha\left(\frac{d^2 f}{d\xi^2} + \frac{d^2 f}{d\xi^2}\right), \qquad (9)$$
$$-c\frac{dg}{d\xi} = A\, f - f^2 g + \alpha\left(\frac{d^2 g}{d\xi^2} + \frac{d^2 g}{d\xi^2}\right) \quad , \qquad (10)$$

where $f(\xi) = u(x,y,t)\ and\ g(\xi) = v(x,y,t)$, obtained upon using the wave variable $\xi = x + y - c\, t$, from the continuity equation $u_x + v_y = 0$, also by using the first integral method we get:

$$\frac{df}{d\xi} + \frac{dg}{d\xi} = 0$$

Integrating, we get

$$f(\xi) + g(\xi) = a \qquad (11)$$

where a is an integration constant, using (11) in (9) and (10) we get

$$f'' = \frac{1}{2\alpha}[f^3 + f(A+1) + B - cf' - af^2] \qquad (12)$$

$$g'' = \frac{1}{2\alpha}[-cg' + Ag - aA - g^3 + 2ag^2 - a^2 g] \qquad (13)$$

Case 1: Using equation (8) in (12) we have the system.

$$X'(\xi) = Y(\xi)$$
$$Y'(\xi) = \frac{1}{2\alpha}[X^3 + X(A+1) + B - cY - aX^2] \qquad (14)$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions of (14), and $F(X,Y) = \sum_{i=0}^{r} a_i(X)\, Y^i$ is an irreducible polynomial in the complex domain $C[X\,,Y\,]$ such that

$$F\big(X(\xi),Y(\xi)\big) = \sum_{i=0}^{r} a_i\big(X(\xi)\big)\, Y^i(\xi) = 0 \qquad (15)$$

where, $a_i(X); (i = 0,1,\ldots,r)$ are polynomial and $a_r(X) \neq 0$. Equation (15) called the first integral method there exist a polynomial $g(X) + h(X)Y$ in the complex domain $C[X\,,Y\,]$ such that:

$$\frac{dF}{d\xi} = \frac{dF}{dX}\frac{dX}{d\xi} + \frac{dF}{dY}\frac{dY}{d\xi} = [g(X) + h(X)Y]\sum_{i=0}^{r} a_i(X)\, Y^i \qquad (16)$$

We consider $r = 1$ in equation (16)

$$a_1' = a_1\, h(X) \qquad (17)$$

$$a_0' - \frac{ca_1}{2\alpha} = h(X)a_0 + g(X)a_1 \qquad (18)$$

$$\frac{a_1}{2\alpha}(X^3 + (A+1)X - B - aX^2) = a_0\, g(X) \qquad (19)$$

Since $a_i(X); (i = 0,1)$ are polynomial, then from (17) we have that a_1 is constant and $h(X) = 0$ for simplicity, take $a_1(X) = 1$, and balancing the degrees of $g(X)$

and $a_0(X)$ we conclude that $deg(g(X)) = 1$, only. Now suppose that $g(X) = A_0X + A_1$ and we obtain from (18) as follows

$$a_0 = \frac{1}{2}A_0X^2 + \left(A_1 + \frac{c}{2\alpha}\right)X + A_2$$

where A_2 is an arbitrary integration constant. Substituting $a_0(X), a_1(X)$ and $g(X)$ in (19) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we get

Family 1:

$$A_0 = \mp\frac{1}{\sqrt{\alpha}}, \qquad A_1 = \frac{-c \mp a\sqrt{\alpha}}{3\alpha}, A_2 = \frac{3B}{2c \mp 2a\sqrt{\alpha}} \tag{20}$$

setting (20) in (15) we have the following ordinary differential equations

$$Y(\xi) = \pm\frac{1}{2\sqrt{\alpha}}X^2 + \left(\frac{6B\alpha + c\left(2c \mp 2a\sqrt{\alpha}\right)}{2\alpha\left(2c \mp 2a\sqrt{\alpha}\right)}\right)X + \frac{3B}{2c \mp 2a\sqrt{\alpha}} \tag{21}$$

If we solve (21) by using (14) and (11), respectively, we have the analytical solutions of the reaction–diffusion Brusselator systems:

$$u(x,y,t) = \frac{\pm 1}{2\alpha a + 2\sqrt{\alpha}c}\left(\beta_0 \tan\left(\frac{\left((x+y-ct)+\xi_0\right)\beta_0}{4\alpha\left(\sqrt{\alpha}\,a \pm c\right)}\right) - c^2 - 3\alpha B - \sqrt{\alpha}ca\right),$$

(22)

$$v(x,y,t) = a \mp \frac{1}{2\alpha a + 2\sqrt{\alpha}c}\left(\beta_0 \tan\left(\frac{\left((x+y-ct)+\xi_0\right)\beta_0}{4\alpha\left(\sqrt{\alpha}\,a \pm c\right)}\right) - c^2 - 3\alpha B - \sqrt{\alpha}ca\right),$$

where

$\beta_0 = \sqrt{-\alpha c^2(a^2+6B)+3\alpha^2 B(4a-3B) \mp 6(a-2)\alpha^{\frac{3}{2}} Bc \mp 2a\sqrt{\alpha}c^3 - c^4}$ and ξ_0 is an arbitrary integration constant.

Family 2:

$$A_0 = \mp \frac{1}{\sqrt{\alpha}}\ , \qquad A_1 = \frac{-c \mp a\sqrt{\alpha}}{3\,\alpha}\ , A_2 = \pm\beta_1 \qquad (23)$$

such that $\beta_1 = \frac{9\,\alpha(A+1)+3c\left(c \mp a\sqrt{\alpha}\right)-2\alpha\,a^2 \mp 4a\,c\sqrt{\alpha}-2\,c^2}{18\,\alpha^{\frac{3}{2}}}$ setting (23) in (15) we have the following ordinary differential equations

$$Y(\xi) = \pm\frac{1}{2\sqrt{\alpha}}X^2 + \left(\frac{6B\alpha + c\left(2c \mp 2\,a\sqrt{\alpha}\right)}{2\alpha\left(2c \mp 2\,a\sqrt{\alpha}\right)}\right)X \pm \beta_1 \qquad (24)$$

If we solve (24) by using (14) and (11), respectively, we have the analytical solutions of the reaction–diffusion Brusselator systems:

$$u(x,y,t) = \left(\frac{\pm 1}{2c+2a\sqrt{\alpha}}\right)\left(\beta_2\tan\frac{\beta_2(x+y-ct+\xi_1)}{4\left(a\,\alpha \pm \sqrt{\alpha}\,c\right)} - 9a\sqrt{\alpha}c - 27\alpha B - 9c^2\right),$$

$$(25)$$

$$v(x,y,t) = a \mp \left(\frac{1}{2c \pm 2a\sqrt{\alpha}}\right)\left(\beta_2\tan\frac{\beta_2(x+y-ct+\xi_1)}{4\left(a\,\alpha \pm \sqrt{\alpha}\,c\right)} - 9a\sqrt{\alpha}c - 27\alpha B - 9c^2\right)$$

where

$\beta_2 =$

$\sqrt{8a^2\alpha^{\frac{3}{2}}\beta_1 + c^2\left(-81\alpha a^2 + 8\sqrt{\alpha}\beta_1 - 486\alpha B\right) + ac\left(16\alpha\beta_1 - 486\alpha^{\frac{3}{2}}B\right) - 162a\sqrt{\alpha}c^3 - 729\alpha^2 B^2 - 8}$

and ξ_1 is an arbitrary integration constant.

Case 2: Using equation (8) in (13) we have the system.

$$G'(\xi) = Q(\xi)$$
$$Q'(\xi) = \frac{-1}{2\alpha}[G^3 + G(a + a^2) + aA - cQ - 2aG^2] \quad (26)$$

According to the first integral method, we suppose that $G(\xi)$ *and* $Q(\xi)$ are the nontrivial solutions of (26), and $H(G,Q) = \sum_{i=0}^{r} b_i(G)\, Q^i$ is an irreducible polynomial in the complex domain $C[G, Q]$ such that

$$H\big(G(\xi), Q(\xi)\big) = \sum_{i=0}^{r} b_i\big(G(\xi)\big)\, Q^i(\xi) = 0 \quad (27)$$

where, $b_i(G); (i = 0,1, \dots, r)$ are polynomial and $b_r(G) \neq 0$. Equation (27) called the first integral method there exist a polynomial $g(G) + h(G)Q$ in the complex domain $C[G, Q]$ such that:

$$\frac{dH}{d\xi} = \frac{dH}{dG}\frac{dG}{d\xi} + \frac{dH}{dQ}\frac{dQ}{d\xi} = [g(G) + h(G)Q]\sum_{i=0}^{r} b_i(G)\, Q^i \quad (28)$$

We consider $r = 1$ in equation (28)

$$b_1' = b_1\, h(G) \quad (29)$$

$$b_0' + \frac{cb_1}{2\alpha} = h(G)b_0 + g(G)b_1 \quad (30)$$

$$\frac{-b_1}{2\alpha}(G^3 + G(a + a^2) + aA - 2aG^2) = b_0\, g(G) \quad (31)$$

Since $b_i(G); (i = 0, 1)$ are polynomial, then from (29) we have that b_1 is constant and $h(G) = 0$ for simplicity, take $b_1(G) = 1$, and balancing the degrees of $g(G)$ and $b_0(G)$ we conclude that $deg(g(G)) = 1$, only. Now suppose that $g(G) = B_0 G + B_1$and we obtain from (30) as follows

$$b_0 = \frac{1}{2}B_0 G^2 + \left(B_1 - \frac{c}{2\alpha}\right) G + B_2$$

where B_2 is an arbitrary integration constant. Substituting $b_0(G), b_1(G)$ and $g(G)$ in (31) and setting all the coefficients of powers G to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we get

Family 1:

$$B_0 = \pm\frac{1}{\sqrt{\alpha}}, \qquad B_1 = \frac{-c \pm 2a\sqrt{\alpha}}{3\,\alpha}, \; B_2 = \frac{3aA}{2c \mp 4\,a\sqrt{\alpha}} \tag{32}$$

setting (32) in (27) we have the following ordinary differential equations

$$Q(\xi) = \pm\frac{1}{2\sqrt{\alpha}}G^2 + \left(\frac{5c \mp 4\,ac\sqrt{\alpha}}{6\alpha}\right)G - \frac{3aA}{2c \mp 4\,a\sqrt{\alpha}} \tag{33}$$

If we solve (33) by using (26) and (11), respectively, we have the analytical solutions of the reaction–diffusion Brusselator systems:

$$v(x,y,t) = \frac{1}{6\sqrt{\alpha}\sqrt{2\sqrt{\alpha}\,a + c}}\left(\beta_3 \tan\left(\frac{\beta_3(x+y-ct+\xi_2)}{12\,\alpha\sqrt{2\sqrt{\alpha}\,a+c}}\right) - c\sqrt{2\sqrt{\alpha}\,a+c}\,\left(4\sqrt{\alpha}\,a \pm 5\right)\right),$$

(34)

$$u(x,y,t) = a - \frac{1}{6\sqrt{\alpha}\sqrt{2\sqrt{\alpha}\,a + c}}\left(\beta_3 \tan\left(\frac{\beta_3(x+y-ct+\xi_2)}{12\,\alpha\sqrt{2\sqrt{\alpha}\,a+c}}\right) - c\sqrt{2\sqrt{\alpha}\,a+c}\,\left(4\sqrt{\alpha}\,a \pm 5\right)\right),$$

where $\beta_3 = \sqrt{-108a\,\alpha^{\frac{3}{2}}A + c^3\left(-\left(4\sqrt{\alpha}\,a \pm 5\right)^2\right) - 2a\sqrt{\alpha}\,c^2\left(4\sqrt{\alpha}\,a \pm 5\right)^2}$ and ξ_2 is an arbitrary integration constant.

Family 2:

$$B_0 = \pm\frac{1}{\sqrt{\alpha}}, \; B_1 = \frac{-c \pm 2a\sqrt{\alpha}}{3\,\alpha}, \; B_2 = \mp\sqrt{\alpha}\,\frac{9\alpha(a+a^2) - c^2 \mp 2ac\sqrt{\alpha} + 8a^2\alpha}{18\,\alpha^2} \tag{35}$$

setting (35) in (27) we have the following ordinary differential equations

$$Q(\xi) = \pm\frac{1}{2\sqrt{\alpha}}G^2 + \left(\frac{5c \mp 4\,ac\sqrt{\alpha}}{6\alpha}\right)G \mp \sqrt{\alpha}\frac{9\alpha(a+a^2) - c^2 \mp 2ac\sqrt{\alpha} + 8a^2\alpha}{18\,\alpha^2}$$

(36)

If we solve (36) by using (26) and (11), respectively, we have the analytical solutions of the reaction–diffusion Brusselator systems:

$$v(x,y,t) = \frac{1}{12\sqrt{\alpha}}\left(\beta_4\, tan\left(\frac{\beta_4(x+y-ct+\xi_3)}{12\,\alpha}\right) - c\ \ (4\sqrt{\alpha}\,a \mp 5)\right),$$

(37)

$$u(x,y,t) = a - \frac{1}{12\sqrt{\alpha}}\left(\beta_4\, tan\left(\frac{\beta_4(x+y-ct+\xi_3)}{12\,\alpha}\right) - c\ \ (4\sqrt{\alpha}\,a \mp 5)\right)$$

where $\beta_4 = \sqrt{-c^2\left(16\,\alpha\,a^2 \pm 40\sqrt{\alpha}\,a + 29\right) + 4a(17a+9)\alpha + 8a\sqrt{\alpha}\,c}$ and ξ_3 is an arbitrary integration constant.

4. Conclusions

In this paper, the first integral method is used for constructing a wide classes of periodic travelling wave solutions of Reaction–Diffusion Brusselator systems in two-dimensional system. It was proved that the method is a powerful mathematical technique for investigating and finding the exact solutions for the partial differential equations.

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