Modules With the δ-coclosed Intersection Property

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Abstract. In this article we present and investigate the notion of modules which the intersection of any two δ-coclosed submodules in it is again δ-coclosed, such modules, namely, modules with the δ-coclosed intersection property. Many basic properties of δ-coclosed submodules, and modules with the δ-coclosed intersection property are given in this work.

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1. Introduction

Throughout this article, all rings are associative with identity and all modules are unitary right R-modules, unless otherwise stated. A submodule N of a module M is said to be essential (briefly $N \leq_e M$) if $N \cap K \neq 0$ for any nonzero submodule K of M [4]. For an R-module M, the set $Z(M) = \{m \in M \mid ann_R(m) \leq_e R\}$ is called a singular submodule. A module M is called singular if $Z(M) = M$, and it is called nonsingular if $Z(M) = 0$ [4]. A submodule N of a module M is called C-singular (briefly $N \leq_{cs} M$) if M/N is a singular module. A submodule N of M is called δ-small in M (briefly $N \ll_s M$) if for any C-singular submodule K of M with $N + K = M$ implies $K = M$ [8]. For $N, K \leq M$, N is called δ-coessential of K in M (briefly $N \leq_{sce} K$ in M) if K/N is δ-small in M/N [6]. Clearly, every small submodule is δ-small, and hence every coessential submodule N of K in M is δ-coessential of K in M. A submodule N of M is called δ-coclosed (briefly $N \leq^{scc} M$) if, whenever N/K is singular and N/K is δ-small in M/K for some $K \leq N$ implies $K = N$ [3]. In this work we introduce and investigate the concept of modules with the δ-coclosed intersection property, where an *R*-module *M* is called module has δ-coclosed inte-rsection property if the intersection of any two δ-coclosed submodules of *M* is again δ-coclosed (briefly δ-CCIP). This paper is structured in two sections. In section 1, we stated some well-known properties of C-singular, δ-small, δ-coessential and δ-coclosed submodules, which needed in this work. In section 2, we present several general properties of modules with δ-CCIP. We prove that for an *R*-module *M*, if *M* has the δ-CCIP, then for any decomposition $M = A \oplus B$ and for all $\phi \in Hom_R(A,B)$, $Ker\phi$ is δ-coclosed in *M*. We will denotes $ann_R(M) = \{r \in R |\ rm = 0 \text{ for all } m(\neq 0) \in M\}$.

Before anything, we will list some known properties of C-singular, δ-small, δ-coessential and δ-coclosed submodules respectively.

Lemma 1.1 [4] Let *M* be an *R*-module and $N \le M$. If $N \le_e M$ then $N \le_{cs} M$. The converse is true, whenever *M* is nonsingular.

Lemma 1.2 [8] Let *M* be an *R*-module. Then the following hold.

(*a*) Let $K \le N$ and *L* are submodules of a module *M*.

(*i*) If $K \ll_s N$, then $K \ll_s M$.

(*ii*) If $N \ll_s M$, then $K \ll_s M$.

(*iii*) $N \ll_s M$ if and only if $K \ll_s M$ and $N/K \ll_s M/K$.

(*iv*) $N + L \ll_s M$ if and only if $N \ll_s M$ and $L \ll_s M$.

(*b*) If $K \ll_s M$ and $\varphi : M \to N$ is a homomorphism, then $\varphi(K) \ll_s N$.

(*c*) If $K_i \le M_i \le M$ for $(i = 1,2)$, and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_s M_1 \oplus M_2$ if and only if $K_1 \ll_s M_1$ and $K_2 \ll_s M_2$.

Lemma 1.3 [5] Let *M* be a uniform (or singular) *R*-module and $N \subset M$. Then $N \ll M$ if and only if $N \ll_s M$.

Lemma 1.4 [5] Let M, N be an *R*-modules and let $A \le B \le C \le M$. Then

(*i*) $B \le_{sce} C$ in M if and only if $B/A \le_{sce} C/A$ in M/A.

(*ii*) $A \le_{sce} C$ in M if and only if $A \le_{sce} B$ and $B \le_{sce} C$ in *M*.

(*iii*) Let $\varphi : M \to N$ be an epimorphism. If $A \le_{sce} B$ in M then $\varphi(A) \le_{sce} \varphi(B)$ in *N*.

Lemma 1.5 [5] Let M be an R-module, and let $A \leq_{sce} B$ in M. Then

(i) If $C \leq_{sce} D$ in M, then $A+C \leq_{sce} B+D$ in M.

(ii) If $X \leq M$, then $A+X \leq_{sce} B+X$ in M. The converse is true if, $X \ll_s M$.

(iii) If $X \ll_s M$, then $A \leq_{sce} B+X$ in M.

A module M is called δ-supplemented if, for every submodule L of M, there exists a submodule K of M such that $M = L+K$ and $L \cap K \ll_s K$. In this case K is called δ-supplement of L in M [6]. A module M is called weakly δ-supplemented if, for every submodule L of M, there exists a submodule K of M such that $M = L+K$ and $L \cap K \ll_s M$. In this case K is called a weak δ-supplement of L in M [3].

Lemma 1.6 [3] For an R-module M and a submodule N of M. Consider the following statements.

(i) N is a δ-supplement submodule in M.

(ii) N is a δ-coclosed submodule in M.

(iii) For all $X \leq N$ and $X \ll_s M$ implies $X \ll_s N$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$. If N has a weak δ-supplement in M, then (i) through (iii) are equivalent.

2. Modules with the δ-coclosed Intersection Property

We begin by the following Proposition.

Proposition 2.1 Every direct summand of a module is δ-coclosed.

Proof. Let M be an R-module and let N be a direct summand of M. For some $K \leq M$, $M = N \oplus K$. To prove that N is a δ-coclosed submodule, let $X \leq N$ with N/X singular and $N/X \ll_s M/X$, also $M/X = (N \oplus K)/X = N/X + (K+X)/X$, so by 3rd isomorphism theorem, $(M/X)/(K+X)/X \cong M/(K+X)$. We claim that $M/(K+X)$ is singular. For any $m \in M$, $m = n+k$ for some $n \in N$, $k \in K$. Now, let $r \in ann_R(n+X)$, $nr + X = X$; that is $nr \in X$, but $(m+K+X)r = mr+K+X = nr+kr+K+X = K+X$, so $ann_R(n+X) \subseteq ann_R(m+K+X)$. Since N/X is singular, $ann_R(n+X) \leq_e R$, so $ann_R(m+K+X) \leq_e R$, hence $M/(K+X)$ is singular. But $N/X \ll_s M/X$, so $M/X = (K+X)/X$, then $K+X = M = K+N$. To prove that $X = N$, assume $x \in N \subseteq M$, $x = a+b$ for some $a \in K$, $b \in X \subseteq N$ then $x-b = a \in N \cap K = 0$, thus $x-b = 0$, $x = b \in X$, then $N \subseteq X$ and hence $N = X$. □

Now, we present the following example.

Example 2.2 Let $M = Z \oplus Z_2$ as Z-module, $A = (1,\bar{0})Z$ and $B = (1,\bar{1})Z$ are direct summands of *M*, so A and B are δ-coclosed submodules of M with $A \cap B = 2Z \oplus (\bar{0})$. Take $L = 4Z \oplus (\bar{0})$, $(A \cap B)/L = 2Z \oplus (\bar{0}) \big/ 4Z \oplus (\bar{0})$ is singular, because for $t \in ann_R(m+L)$, where $m \in A \cap B$, then $mt \in L = 4Z \oplus (\bar{0})$ which implies that $t \in 2Z$ and so that $ann_R(m+L) = 2Z \leq_e Z$, hence $(A \cap B)/L$ is singular. On the other hand, $(A \cap B)/L$ is δ-small in M/L but $A \cap B \neq L$, thus $A \cap B$ is not δ-coclosed in *M*.

This example leads us to introduce the following.

Definition 2.3 A module *M* is said to have the δ-coclosed intersection property (briefly δ-CCIP) if, the intersection of any two δ-coclosed submodules of *M* is again δ-coclosed.

The following Proposition, gives some properties of δ-coclosed submodules.

Proposition 2.4 Let *M* be an *R*-module and let $K \leq L \leq M$ .Then the following assertions hold.

(*i*) If $L \leq^{scc} M$, then $L/K \leq^{scc} M/K$.

(*ii*) If $L \leq^{scc} M$ and $N \ll_s M$, then $L+N/N \leq^{scc} M/N$.

(*iii*) If $K \leq^{scc} M$, then $K \leq^{scc} L$. The converse hold whenever $L \leq^{scc} M$.

Proof. (*i*) Let $N/K \leq_{sce} L/K$ in M/K and $(L/K)/(N/K)$ which is isomorphic to L/N is singular, so by Lemma 1.4, $N \leq_{sce} L$, also $N \leq_{cs} L$ which implies $N = L$ and hence $N/K = L/K$.

(*ii*) Suppose that $X/N \leq_{sce} (L+N)/N$ in M/N and $(L+N/N)/(X/N) \cong L+N/X$ is singular, so by Lemma 1.4, $X \leq_{sce} L+N$, also $(L+N)/X$ is singular. Clearly, $X = (L \cap X)+N$, thus $(L \cap X)+N \leq_{sce} L+N$, but $N \ll_s M$, so by Lemma 1.5, $(L \cap X) \leq_{sce} L$. Moreover, we have $L/(L \cap X)$ is singular, to prove this: for all $(l+n)+X \in L+N/X$, we have $ann_R[(l+n)+X] \leq_e R$, so $ann_R(l+X) \leq_e R$. But $ann_R(l+X) \subseteq ann_R(l+L \cap X)$, hence $ann_R(l+L \cap X) \leq_e R$ for all $l \in L$; that is $L/L \cap X$ singular. But $L \leq^{scc} M$, so $L \cap X = L$, thus $X = (L \cap X)+N = L+N$. Therefore $X/N = (L+N)/N$.

(*iii*) Assume that $X \leq_{sce} K$ in L and $X \leq_{cs} K$, so $K/X \ll_s L/X$ and K/X is singular. Thus $K/X \ll_s M/X$ and K/X is singular; that is $X \leq_{sce} K$ in M and $X \leq_{cs} K$. But $K \leq^{scc} M$, so $K = X$ and hence $K \leq^{scc} L$. Conversely, let $L \leq^{scc} M$. Assume that $K/X \ll_s M/X$ and K/X is singular. Since $X \leq L$, then by (*i*), $L/X \leq^{scc} M/X$ and hence by Lemma 1.6, $K/X \ll_s L/X$, but $K \leq^{scc} L$, thus $K = X$. □

Lemma 2.5 Let M and N be R-modules with $\varphi : M \to N$ be an R-epimorphism. If M is weakly δ-supplemented, then N also is weakly δ-supplemented .

Proof. Let $X \le N$, then $\varphi^{-1}(X) \le M$. Since M is weakly δ-supplemented, so there exists $B \le M$ such that $\varphi^{-1}(X) + B = M$ and $\varphi^{-1}(X) \cap B \ll_s M$, then $X + \varphi(B) = N$ and $X \cap \varphi(B) \ll_s N$, this proving that N is weakly δ-supplemented. □

Corollary 2.6 If M is a weakly δ-supplemented module and $B \le M$, then M/B is also weakly δ-supplemented.

Proof. It follow directly by taking the natural epimorphism $\pi : M \to M/B$. □

Note: If M and M' are two R-modules, $A \le M$ and $B \le M'$ such that $A \ll_s M$ with $A \cong B$ then it is not necessary that $B \ll_s M'$, as example: consider the Z-modules Z, Q and let $A = Z \le Q$, $B = Z \le Z$. It is clear that $A = Z$ is a small submodule of Q, so it is δ-small in Q with $A \cong B$, but $B = Z$ is not δ-small in Z.

However, we consider the following condition (t) for R-modules M, M':

- If $A \le M$ and $B \le M'$ such that $A \ll_s M$ with $A \cong B$ implies $B \ll_s M'$ …(t)

Proposition 2.7 Let M be a weakly δ-supplemented R-module which satisfying condition (t), and $B \le C$ are submodules of M. If $C/B \le^{scc} M/B$ and $B \le^{scc} M$, then $C \le^{scc} M$.

Proof. Since M is a weakly δ-supplemented module, then by Corollary 2.6, M/B is also weakly δ-supplemented. Since $C/B \le^{scc} M/B$ and $B \le^{scc} M$, then by Lemma 1.6, C/B and B are δ-supplement submodules in M/B and M respectively. We have to show that C is a δ-supplement in M. Suppose C/B is a δ-supplement of C'/B in M/B and B is a δ-supplement of B' in M. Then $M/B = C/B + C'/B$ and $C/B \cap C'/B = (C \cap C')/B \ll_s C/B$, also $M = B + B'$ such that $B \cap B' \ll_s B$. Clearly, $B \cap B' \ll_s C$. We claim that C is a δ-supplement of $B' \cap C'$ in M. To prove this assertion, notice that $M = (C \cap C') + B'$ and $M = C + C'$, so by [2, Lemma 1.1.6], $M = C + (B' \cap C')$. Now, $C = C \cap (B + B') = B + (C \cap B')$ and $C \cap C'/B \ll_s C/B$, but $M = B + B'$, so by [7, Lemma 3.15], $\dfrac{(C \cap C') \cap B'}{B \cap B'} \cong \dfrac{C \cap C'}{B}$. By condition (t), $\dfrac{(C \cap C') \cap B'}{B \cap B'} \ll_s \dfrac{C}{B \cap B'}$ but $B \cap B' \ll_s C$ implies $(C \cap C') \cap B' \ll_s C$, thus C is a δ-supplement of $B' \cap C'$ in M and hence by Lemma 1.6, C is δ-coclosed in M. □

Next we shall give some results about modules with the δ-CCIP, but first we need the following.

Definition 2.8 A δ-coclosure of a submodule *B* of a module *M* is a δ-coessential submodule of *B* in *M* which is also a δ-coclosed submodule of *M*.

Notice, a δ-coclosure of a submodule of a module may not always exists, also if it exists then it is not unique, for example: in *Z*-module *Z*, *2Z* has no δ-coclosure.

Definition 2.9 A module for which every submodule has a unique δ-coclosure is called a unique δ-coclosure module or UδCC.

Theorem 2.10 Let *M* be a singular *R*-module and $A \leq M$. Then *A* is δ-coclosed in *M* if and only if *A* is coclosed in *M*.

Proof. Let $A \leq^{scc} M$. To prove $A \leq^{cc} M$, assume $B \leq A$ and $A/B \ll M/B$, then $A/B \ll_s M/B$. Since *M* is singular, so *A* is singular and hence A/B is singular. Thus, we have $B \leq_{sce} A$ in M and $B \leq_{cs} A$, but $A \leq^{scc} M$ these implies $A = B$. Conversely, let $L \leq A$ such that A/L is singular and $A/L \ll_s M/L$. Since M is a singular module, so M/L is also singular and hence by Lemma 1.3, $A/L \ll M/L$. But $A \leq^{cc} M$, thus $A = L$. □

Corollary 2.11 Let *M* be a singular *R*-module and let $A \leq B \leq M$. Then *A* is a δ-coclosure of *B* in *M* if and only if *A* is a coclosure of *B* in *M*.

Corollary 2.12 Let *M* be a singular *R*-module. Then *M* has the δ-CCIP if and only if *M* has the CCIP.

Theorem 2.13 Let *M* be a uniform *R*-module, $N \leq M$. Then *N* is δ-coclosed in *M* if and only if *N* is coclosed in *M*.

Proof. If $N \leq^{scc} M$. Suppose $A \leq N$ such that $N/A \ll M/A$, so $N/A \ll_s M/A$. By Lemma 1.1, M/A is singular, so N/A is singular, this mean *A* is δ-coessential of *N* in *M* and *A* is C-singular of *N*, but $N \leq^{scc} M$ thus $A = N$. Conversely, assume that $K \leq N$ such that N/K singular and $N/K \ll_s M/K$. Since $K \leq M$, then $K \leq_e M$, again by Lemma 1.1, M/K is singular. On the other hand, small and δ-small are coincide in a singular module, by Lemma 1.3. So, we have $N/K \ll M/K$, but $N \leq^{cc} M$ so this implies $K = N$. □

By applying the previous Theorem, in the *Z*-module *Z*, the coclosed and the δ-coclosed submodules are coincide, and hence *Z* as *Z*-module has the δ-CCIP.

Corollary 2.14 Let *M* be a uniform *R*-module and let $L \leq N \leq M$. Then *L* is a δ-coclosure of *N* in *M* if and only if *L* is a coclosure of *N* in *M*.

Corollary 2.15 Let *M* be a uniform *R*-module. Then *M* has the δ-CCIP if and only if *M* has the CCIP.

Proposition 2.16 Let *M* be an *R*-module. If *M* has the δ-CCIP, then for every decomposition $M = A \oplus B$ and for all $\phi \in Hom_R(A, B)$, $Ker\phi$ is δ-coclosed in *M*.

Proof. Assume $M = A \oplus B$ has the δ-CCIP and $\phi : A \to B$ is an *R*-homomorphism. Consider the set $S = \{a + \phi(a) : a \in A\}$. It is easy to see that $M = S \oplus B$, as follows: let $x \in M$ then $x = a + b$ where $a \in A$, $b \in B$. Now, we can say $x = a + \varphi(a) - \varphi(a) + b \in S + B$, thus $M = S + B$. Let $p \in S \cap B$, $p = a + \varphi(a)$ where $a \in A$, $p \in B$ then $a = p - \varphi(a) \in A \cap B = 0$, so $a = 0$, implies $p = \varphi(a) = 0$, thus $S \cap B = 0$ and hence $M = S \oplus B$. So each of A and S is a direct summand of *M* and hence A, S are δ-coclosed submodules of *M*. Therefore $Ker\phi = A \cap S$ is δ-coclosed in *M*. □

The following Proposition and Corollary give a characterization for modules with the δ-CCIP.

Proposition 2.17 Let *M* be an *R*-module. Then *M* has the δ-CCIP if and only if for any $N \leq^{scc} M$, *N* has the δ-CCIP.

Proof. If *M* has the δ-CCIP. Suppose that K_1, K_2 are two δ-coclosed submodules of *N*, but $N \leq^{scc} M$, so by Proposition 2.4 (iii), K_1 and K_2 are δ-coclosed in M, hence $K_1 \cap K_2 \leq^{scc} M$. Since $K_1 \cap K_2 \leq N \leq M$, again by Proposition 2.4 (iii), $K_1 \cap K_2 \leq^{scc} N$. Conversely, it follows by taking $N = M$. □

Corollary 2.18 Let *M* be an *R*-module. Then *M* has the δ-CCIP if and only if every direct summand of *M* has the δ-CCIP.

Remark 2.19 The direct sum of modules with the δ-CCIP may not have the δ-CCIP, see Example 2.2.

The next Proposition give a condition under which the direct sum of modules with the δ-CCIP is again has the δ-CCIP. Before this result, we give the following Lemma.

Lemma 2.20 Let $M = M_1 \oplus M_2$, where M_1, M_2 be two *R*-modules with $ann_R M_1 + ann_R M_2 = R$. Then a submodule *K* of *M* is δ-coclosed if and only if, there exists δ-coclosed submodules K_1 of M_1, and K_2 of M_2 such that $K = K_1 \oplus K_2$.

Proof. Assume *K* is δ-coclosed of $M = M_1 \oplus M_2$ with $ann_R M_1 + ann_R M_2 = R$, so by [1, Prop 4.2], we get $K = K_1 \oplus K_2$ for some $K_1 \leq M_1$ and $K_2 \leq M_2$. By Proposition 2.4, we have $K_2 \cong K/K_1 \leq^{scc} M/K_1$, but $M/K_1 = M_1 \oplus M_2 / K_1 \oplus (0) \cong (M_1/K_1) \oplus M_2$ and $K_2 \leq M_2$, so by Proposition 2.4 (iii), $K_2 \leq^{scc} M_2$. Similarly, $K_1 \leq^{scc} M_1$. Conversely, let $L \leq_{cs} K$ and $L \leq_{sce} K$, to prove that $L = K$, where $K = K_1 \oplus K_2$ and K_1, K_2 are δ-coclosed submodules of M_1, M_2

respectively. Since $L \le M$ and $ann_R M_1 + ann_R M_2 = R$, then $L = L_1 \oplus L_2$ for some $L_1 \le M_1, L_2 \le M_2$. But K/L is singular ,so $\frac{K_1 \oplus K_2}{L_1 \oplus L_2} = \frac{K_1}{L_1} \oplus \frac{K_2}{L_2}$ is singular, thus each of $K_1/L_1, K_2/L_2$ is singular; that is $L_1 \le_{cs} K_1$ and $L_2 \le_{cs} K_2$. On the other hand, $K/L \ll_s M/L$, thus $\frac{K_1}{L_1} \oplus \frac{K_2}{L_2} = \frac{K_1 \oplus K_2}{L_1 \oplus L_2} \ll_s \frac{M_1 \oplus M_2}{L_1 \oplus L_2} = \frac{M_1}{L_1} \oplus \frac{M_2}{L_2}$, so by Lemma 1.2, $K_1/L_1 \ll_s M_1/L_1$ and $K_2/L_2 \ll_s M_2/L_2$; that is $L_1 \le_{sce} K_1$ in M_1 and $L_2 \le_{sce} K_2$ in M_2, but both of K_1 and K_2 is δ-coclosed in M_1, M_2 respectively , so $L_1 = K_1$, also $L_2 = K_2$, and hence $L = K$. □

Theorem 2.21 Let $M = M_1 \oplus M_2$ be an *R*-module such that $ann_R M_1 + ann_R M_2 = R$. Then *M* has the δ-CCIP, whenever M_1 and M_2 has the δ-CCIP.

Proof. Suppose L_1, L_2 are two δ-coclosed submodules of $M = M_1 \oplus M_2$, then by Lemma 2.20, $L_1 = A_1 \oplus B_1$, $L_2 = A_2 \oplus B_2$ for some A_1, A_2 are δ-coclosed in M_1 and B_1, B_2 are δ-coclosed in M_2, so by δ-CCIP of M_1 and M_2 respectively, we get $A_1 \cap A_2 \le^{scc} M_1$ and $B_1 \cap B_2 \le^{scc} M_2$. Thus again by Lemma 2.20, we get $(A_1 \cap A_2) \oplus (B_1 \cap B_2) \le^{scc} M_1 \oplus M_2$; that is $L_1 \cap L_2 \le^{scc} M$ and hence M has the δ-CCIP. □

Proposition 2.22 Let *M* be a weakly δ-supplemented *R*-module which satisfying condition (t). Then *M* has the δ-CCIP if and only if M/A has the δ-CCIP, for all $A \le^{scc} M$.

Proof. If *M* has the δ-CCIP. Assume L_1/A and L_2/A are δ-coclosed submodules of M/A. Since $A \le^{scc} M$, so by Proposition 2.7, L_1 and L_2 are δ-coclosed submodules of *M*, but *M* has the δ-CCIP, then $L_1 \cap L_2 \le^{scc} M$ and so by Proposition 2.4, we have $(L_1 \cap L_2)/A \le^{scc} M/A$; that is $(L_1/A) \cap (L_2/A) \le^{scc} M/A$. Hence M/A has the δ-CCIP, for all $A \le^{scc} M$. Conversely, it follows by taking $A = 0$. □

By combining Proposition 2.17, Corollary 2.18 and Proposition 2.22, we get the following result.

Proposition 2.23 Let *M* be a weakly δ-supplemented *R*-module which satisfying condition (t). Then the following statements are equivalent.

(*i*) *M* has the δ-CCIP.

(*ii*) For all $A \le^{scc} M$, A has the δ-CCIP.

(*iii*) For all $A \le^{\oplus} M$, A has the δ-CCIP.

(*iv*) For all $A \le^{scc} M$, M/A has the δ-CCIP.

Now, we turn our attention to the behavior of modules with δ-CCIP under localization. However we need the following Lemmas.

Lemma 2.24 Let M be an R-module and let S be a multiplicative closed subset of R, then $S^{-1}(Z(M)) \subseteq Z(S^{-1}M)$. The reverse inclusion hold if, M is singular.

Proof. Let $\lambda \in S^{-1}(Z(M))$, so there exists $m \in Z(M)$, $s \in S$ such that $\lambda = m/s$. Since $m \in Z(M)$, $ann_R(m) \leq_e R$, but it is clear that $ann_R(m) \subseteq ann_R(m/s)$, so $ann_R(\lambda) \leq_e R$, thus $\lambda \in Z(S^{-1}M)$ and so that $S^{-1}(Z(M)) \subseteq Z(S^{-1}M)$.Conversely, if M is singular; that is $Z(M) = M$, then $S^{-1}M \subseteq Z(S^{-1}M)$. On the other hand, we have $Z(S^{-1}M) \subseteq S^{-1}(M)$. Thus $Z(S^{-1}M) = S^{-1}M$ and hence $S^{-1}M$ is a singular as R-module. □

Lemma 2.25 Let M be an R-module, $N \leq M$ and let S be a multiplicative closed subset of R. Consider $S^{-1}M$ as an R-module. If N is C-singular in M, then $S^{-1}N$ is C-singular in $S^{-1}M$.

Proof. As N is C-singular, M/N is singular, hence by previous Lemma, $S^{-1}(M/N)$ is singular. But $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$, thus $S^{-1}M/S^{-1}N$ is singular and hence $S^{-1}N$ is C-singular in $S^{-1}M$. □

Now, we consider the following condition (c) for an R-module M.

- For $L \leq N \leq M$, $S^{-1}L$ is C-singular in $S^{-1}N$ implies L is C-singular in N…(c)

Lemma 2.26 Let M be an R-module, $N \leq M$ and let S be a multiplicative closed subset of R such that condition (c) hold. Then N is δ-coclosed in M as R-module if and only if $S^{-1}N$ is δ-coclosed in $S^{-1}M$ as R-module, provided $S^{-1}A = S^{-1}B$ iff $A = B$ for all $A, B \leq M$.

Proof. Let $N \leq^{scc} M$. Assume $S^{-1}L \leq_{cs} S^{-1}N \leq S^{-1}M$ and $\frac{S^{-1}N}{S^{-1}L} \ll_s \frac{S^{-1}M}{S^{-1}L}$. By condition (c), $L \leq_{cs} N$. We can prove $\frac{N}{L} \ll_s \frac{M}{L}$, to see this: let $\frac{N}{L} + \frac{W}{L} = \frac{M}{L}$, where $\frac{M/L}{W/L} \cong \frac{M}{W}$ is singular; that is $W \leq_{cs} M$. By previous Lemma, $S^{-1}W \leq_{cs} S^{-1}M$; that is $\frac{S^{-1}M}{S^{-1}W}$ singular. On the other hand, $\frac{S^{-1}N}{S^{-1}L} + \frac{S^{-1}W}{S^{-1}L} = S^{-1}(\frac{N}{L}) + S^{-1}(\frac{W}{L}) = S^{-1}(\frac{N}{L} + \frac{W}{L}) = S^{-1}(\frac{M}{L}) = \frac{S^{-1}M}{S^{-1}L}$,but $\frac{S^{-1}M/S^{-1}L}{S^{-1}W/S^{-1}L} \cong \frac{S^{-1}M}{S^{-1}W}$ is singular and $\frac{S^{-1}N}{S^{-1}L} \ll_s \frac{S^{-1}M}{S^{-1}L}$, hence $\frac{S^{-1}W}{S^{-1}L} = \frac{S^{-1}M}{S^{-1}L}$, so $S^{-1}W = S^{-1}M$ and by hypothesis $W = M$, thus $\frac{W}{L} = \frac{M}{L}$. Therefore $\frac{N}{L} \ll_s \frac{M}{L}$, but $N \leq^{scc} M$, so $N = L$ and hence $S^{-1}N = S^{-1}L$. Conversely, assume that $S^{-1}N$ is δ-coclosed in $S^{-1}M$ as R-module. Let $A \leq_{cs} N$ and $\frac{N}{A} \ll_s \frac{M}{A}$, thus by previous Lemma, $S^{-1}A \leq_{cs} S^{-1}N$, also we can prove $\frac{S^{-1}N}{S^{-1}A} \ll_s \frac{S^{-1}M}{S^{-1}A}$ as follows: let $\frac{S^{-1}W}{S^{-1}A} \leq_{cs} \frac{S^{-1}M}{S^{-1}A}$ such that $\frac{S^{-1}N}{S^{-1}A} + \frac{S^{-1}W}{S^{-1}A} = \frac{S^{-1}M}{S^{-1}A}$, thus $\frac{S^{-1}M/S^{-1}A}{S^{-1}W/S^{-1}A} \cong \frac{S^{-1}M}{S^{-1}W}$ is singular; that is $S^{-1}W \leq_{cs} S^{-1}M$ and so by condition (c), $W \leq_{cs} M$, thus $\frac{M}{W}$ is singular. Also, we have $\frac{S^{-1}N + S^{-1}W}{S^{-1}A} = \frac{S^{-1}(N+W)}{S^{-1}A} = \frac{S^{-1}M}{S^{-1}A}$, so $S^{-1}(N+W) = S^{-1}M$, and by hypothesis, $N + W = M$, so $\frac{N}{A} + \frac{W}{A} = \frac{N+W}{A} = \frac{M}{A}$. But, $\frac{N}{A} \ll_s \frac{M}{A}$ and $\frac{M/A}{W/A} = \frac{M}{W}$ is singular, so $\frac{W}{A} = \frac{M}{A}$ which implies $\frac{S^{-1}W}{S^{-1}A} = \frac{S^{-1}M}{S^{-1}A}$. Thus $\frac{S^{-1}N}{S^{-1}A} \ll_s \frac{S^{-1}M}{S^{-1}A}$, but $S^{-1}N \leq^{scc} S^{-1}M$, so $S^{-1}N = S^{-1}A$. By hypothesis, we get $N = A$. □

Proposition 2.27 Let M be an R-module and S be a multiplicative closed subset of R such that condition (c) hold. Then M has the δ-CCIP as R-module if and only if $S^{-1}M$ has the δ-CCIP as R-module, provided $S^{-1}A = S^{-1}B$ iff $A = B$ for all $A, B \leq M$.

Proposition 2.28 Let M be an R-module such that condition (c) hold. Then M has the δ-CCIP as R-module if and only if M_P has the δ-CCIP as R-module, for all maximal ideal P of R.

Next, we will show that under certain class of modules, an R-module M has δ-CCIP if and only if $S^{-1}M$ (as R-module) has the δ-CCIP, but first we prove the following results.

Lemma 2.29 Let M be a prime R-module and let S be a multiplicative closed subset of R with $(ann_R M) \cap S = \phi$, then $S^{-1}(Z(M)) = Z(S^{-1}M)$.

Proof. By Lemma 2.24, we have $S^{-1}(Z(M)) \subseteq Z(S^{-1}M)$. Let $m/s \in Z(S^{-1}M)$, $m \neq 0$, then $ann_R(m/s) \leq_e R$. We claim that $ann_R(m) \supseteq ann_R(m/s)$, to see this: let $r \in ann_R(m/s)$ then $mr/s = (m/s)r = 0/1$, so there exists $t \in S$ such that $mrt = 0$, thus $rt \in ann_R(m) = ann_R M$. Since M is a prime R-module, either $r \in ann_R M$ or $t \in ann_R M$. If $t \in ann_R M$, so $t \in (ann_R M) \cap S$ which is a contradiction. Thus $r \in ann_R M$, hence $mr = 0$; that is $r \in ann_R(m)$, so $ann_R(m) \supseteq ann_R(m/s)$, then $ann_R(m) \leq_e R$, hence $m \in Z(M)$, so $m/s \in S^{-1}(Z(M))$.Thus $Z(S^{-1}M) \subseteq S^{-1}(Z(M))$, and so the result is obtained. □

Corollary 2.30 Let *M* be an *R*-module, and *N* be a prime submodule of *M*. Let *S* be a multiplicative closed subset of *R*. If $(N:_R M)\cap S=\phi$ then $S^{-1}(Z(\frac{M}{N}))=Z(S^{-1}\frac{M}{N})$.

Corollary 2.31 Let *M* be an *R*-module, and *N* be a prime submodule of *M*. Let *S* be a multiplicative closed subset of *R* with $(N:_R M)\cap S=\phi$. Then $N\leq_{cs} M$ if and only if $S^{-1}N\leq_{cs} S^{-1}M$ as *R*-module, provided $S^{-1}A=S^{-1}B$ iff $A=B$ for all $A,B\leq M$.

Proof. Let $N\leq_{cs} M$, so by Lemma 2.25, $S^{-1}N\leq_{cs} S^{-1}M$ as *R*-module. Conversely, assume that $S^{-1}N\leq_{cs} S^{-1}M$,thus $\frac{S^{-1}M}{S^{-1}N}$ is a singular module; that is $Z(S^{-1}\frac{M}{N})=Z(\frac{S^{-1}M}{S^{-1}N})=\frac{S^{-1}M}{S^{-1}N}=S^{-1}\frac{M}{N}$. On the other hand, by previous Corollary, $S^{-1}(Z(\frac{M}{N}))=Z(S^{-1}\frac{M}{N})$,so we get $S^{-1}(Z(\frac{M}{N}))=S^{-1}\frac{M}{N}$. Thus by hypothesis, we get $Z(\frac{M}{N})=\frac{M}{N}$; that is $\frac{M}{N}$ singular and hence $N\leq_{cs} M$. □

Now we can get our next result.

Corollary 2.32 Let *M* be a fully prime *R*-module (*i.e.* every proper submodule of *M* is prime), and let *S* be a multiplicative closed subset of *R* such that $(N:_R M)\cap S=\phi$, for all proper submodule *N* of *M*. Then *M* has the δ-CCIP as *R*-module if and only if $S^{-1}M$ has the δ-CCIP as *R*-module, provided $S^{-1}A=S^{-1}B$ iff $A=B$ for all $A,B\leq M$.

Proof. By Corollary 2.31, the condition (c) hold and hence the result follows directly by Lemma 2.26. □

Conclusions

The notion of modules with the δ-coclosed intersection property (δ-CCIP) gives many of good basic properties. As an example of main result, we proved that for an *R*-module *M*, if *M* has the δ-CCIP, for any decomposition $M=A\oplus B$ and for all $\phi\in Hom_R(A,B)$, $Ker\phi$ is δ-coclosed in *M*. Several important results about this concept are obtained in this work.

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